The Measurement of Area

Area = \frac{1}{2}b \times h

Area = \frac{1}{2}(a+b) \times h

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Defining Area

Area can be defined as the quantity of surface within a specified closed boundary.

Progression in Learning about Area

In the National Numeracy Strategy’s ‘Framework for Teaching Mathematics’ it is suggested that formal work on area should begin in Year 4. However, it is possible, and desirable, that teachers should offer children experiences involving area prior to such formal work.

1. Pre-measurement Experiences

It is important for children to gain early experiences of measuring area through play and as a natural part of everyday life e.g. finding out how many plastic cups will fit on a tray at break-time, working out how many children can sit on the rug in the home corner of the classroom or how many pictures made by the children will fit on a display board. Such informal experiences lay the foundations for later work.

The Difficulties of Direct Comparison of Areas

When beginning formal measurement for most measures, children would be asked to make direct comparisons of objects as a precursor to using units for indirect comparison. However, for area, direct comparison can be very difficult because area is a two-dimensional measure. For this reason, teachers tend to avoid direct comparison of areas and instead teach direct comparison using less complex measures such as length, mass and capacity (for which direct comparison is less problematic). Consequently, children’s understanding of area is usually less advanced than their understanding of length or capacity, for example.

To illustrate the extent of the difficulties involved, consider the examples below.

Most of us would have no hesitation in deciding that rectangle A below has a larger area than rectangle B:

![Rectangle A and B]

This, however, is a simple case because the shortest sides of each rectangle are the same and therefore this reduces the direct comparison to a one-dimensional task. If we were to try to directly compare the rectangles C and D (below) it is apparent that the task is now much more difficult even though we are still examining relatively simple shapes:

![Rectangle C and D]

Direct comparison of areas whose boundaries have an even greater degree of irregularity (such as E and F below) is virtually impossible if attempted by visual inspection:
The only sure way to attempt such direct comparison is by cutting up one of the objects and rearranging the pieces on top of the other object. While this is just about feasible with awkward areas drawn on paper, a ‘cutting up and rearranging’ strategy is rather counterproductive in many circumstances e.g. when deciding which of two rugs has the larger area.

Thus, most teachers start formal work about area by introducing arbitrary units.

2. Covering Surfaces using Arbitrary Units
At this stage, children are asked to cover surfaces using arbitrary units i.e. units which are not identical in terms of their area, e.g. buttons, leaves.

*Typical activities with arbitrary units*

- covering a table-top with shells, pebbles or leaves;
- printing using paint and a block or sponge onto a large sheet of paper in preparation for a classroom display and counting the number of prints afterwards;
- deciding how many children can stand on a PE mat in the hall;
- deciding how many mosaic tiles are needed to produce a picture;
- counting how many footprints will cover the sand-pit.

3. Covering surfaces using Non-Standard Units (NNS Year 4)
Children should progress to using units of an identical size such as bottle-tops, pennies, playing cards, A4 sheets of paper, newspaper sheets, postcards, tiling mats etc. It is also worth using some irregular-shaped objects such as a set of identical cardboard ‘footprints’ or drawing round the same child’s hand (repeatedly) onto a flat surface. The area of different surfaces can be compared by measuring each using the same non-standard unit.

Two different measurements with footprints. Which is the more accurate method of covering the surface?

The aim should be to compare measurements made by different children using the same unit and to discuss why children obtain different values for the measurement. In so doing, the following key ideas about area can be established:

a) that minimising the size and number of ‘gaps’ between units when covering the surface
is important;
b) that overlapping of units should be avoided;
c) that covering a surface right up to the boundary is difficult because units don’t usually fit exactly yet it is important because it affects the measuring outcome;
d) that some units are easier to use, and more accurate (in terms of the points in (a), (b) and (c) above) - the accuracy can be discussed and assessed in terms of how close children’s measurements are to those of other children performing the same task.

4. Deciding upon a Standard Unit for Area (NNS Year 4)

By honing children’s understanding of the key ideas above, we arrive at the point where the units selected are only those which fit together without gaps (i.e. they tessellate) such as rectangles, squares, equilateral triangles, regular hexagons because the results obtained are perceived to have greater accuracy.

Further refinement enables children to recognise that, in the majority of area measurement tasks, the square is more versatile than the alternatives. Recognition that using a particular sized square for every measurement is desirable completes the transition to using standard units for area. From this point on, the square centimetre is used for measuring.

![Diagram of square centimetre](image)

Note that the abbreviation for square centimetres (cm²) will not make much sense to children unless they have seen the superscript notation in relation to square numbers: 2², 3² etc. .

5. Measuring Area by Counting Squares (NNS Year 4)

Once the square centimetre has been established as the standard unit for measuring area it becomes possible to begin to make 

**indirect comparisons**

between areas just by counting how many centimetre squares will fit within the boundary of each. Initially, this should be done with shapes of the type shown below.

![Counting squares](image)

Children often find it helpful to put the numbers of the counting sequence in the squares as they count them.

This also allows us to teach children about 

**conservation of area**

i.e. that if a shape is divided up, and the parts rearranged in a new configuration, the area remains the same. For example, the rectangle above could be cut up and rearranged to form the ‘cross’ shape while still preserving the same total area. Using tangrams (see resources section) and shape dissection tasks are useful ways of developing children’s understanding of conservation of area.

The ‘counting squares’ method of measuring areas can then be extended to counting both whole and half centimetre squares, mentally combining pairs of half-squares to make a whole.
For larger areas such as the area of a table-top, use can be made of the ‘flats’ from sets of Dienes’ blocks each of which has an area of 100cm$^2$. Any remaining gaps (at the boundary) can be filled with ‘longs’ (which each have an area of 10cm$^2$) or ‘ones’.

**Typical activities using whole and half-squares include:**

- using geoboards and elastic bands to make shapes of a given area.

  Different shapes with an area of 2 cm$^2$

- exploring whether area and perimeter are related
  i.e. (a) for a given perimeter, what areas are possible?
  (b) for a given area, what perimeters are possible?
  Geoboards, geostrips and loops of string with a fixed length are useful for such investigations.
- making animals/birds/patterns etc. using squares and half-squares and measuring the area.

6. Extending the Counting Squares Method for Awkward Areas (NNS Year 5)
The ‘counting squares’ method for determining areas is satisfactory for areas which can be neatly subdivided into centimetre squares and half-squares. A further development is necessary for more awkward areas e.g. the surface area of a leaf. In such cases, either tracing or drawing round the boundary of the area onto squared paper or overlaying an acetate sheet marked with a grid composed of square centimetres is probably the best approach.
The counting squares method is refined by considering how to tackle the part squares which will occur around the boundary. The **convention** adopted is to count as a whole square each square for which more than half of the square lies within the boundary of the area but to ignore squares for which less than half of the square lies within the boundary of the area. Thus, in the example above, square A would be counted as 1 square but square B would be ignored.

To explain this to children, we use the conservation of area argument of being able to rearrange parts of the leaf while still preserving the overall area. So, in the example above, if we mentally combine the parts of the leaf lying within squares A and B we would have approximately the equivalent of 1 square’s worth of leaf. It is important for children to recognise that measurements can only be **approximate**. It is worth asking different children to measure the same areas and compare answers to illustrate this point.

**Typical activities for the extended counting squares method include:**
- measuring shadows of children’s profiles drawn on paper;
- counting squares in ‘bubble’ letters or numbers;
- measuring how much of a picture is coloured in a particular colour and producing some data about the picture;
- estimating the surface area of a person by tracing round the person onto squared paper;
- measuring the size of a repeating pattern e.g. on wallpaper, wrapping paper, fabric.

### 7. The Area of a Rectangle (NNS Year 5)

The first such area formula that children should encounter is that for a rectangle. It is best derived by using squared paper and drawing a series of rows of a given number of squares:

![Rectangle Diagram]

and 3 rows of 5 squares $= 5 + 5 + 5 = 3 \times 5 = 15$ squares.

Thus, we arrive at the formula:

**The area of a rectangle** $= \text{length} \times \text{breadth}$ or $A = l \times b$.

This is effectively the same argument used in calculating the size of a rectangular array by multiplication and, as such, is better applied to measuring areas after it has been met in lessons about multiplication. Children will usually need to try this approach with several examples in order to see that this applies to rectangles of all sizes. The actual statement of the formula as $A = l \times b$ clearly requires that children be familiar with the use of symbols in writing general statements (probably from work on pattern and early algebra). It is usually best for children to be introduced to the formula written in words before turning this into the shorter, symbolic form.
**Typical activities include:**

- measuring the relative sizes of advertisements in a magazine;
- comparing the areas within a newspaper devoted to different categories of material e.g. news, sport, fashion;
- measuring the size of the hall floor using a trundle wheel and then applying the formula;
- deciding how many cars will fit into the school car park.

**8. Introducing Further Units for Area**

The National Numeracy Strategy ‘Framework for Teaching Mathematics’ recommends that children be introduced to other units for area, namely mm² and m², in Year 5. They should also be expected to develop an appreciation of the approximate size of these units in order that they might be able to estimate these units and to recognise when a particular unit is suitable for measuring a given area.

Using relationships between units of length (1 cm = 10 mm, 100cm = 1 m) we can derive relationships between units of area. Thus:

\[
\begin{align*}
1 \text{cm}^2 &= 10 \text{mm} \times 10 \text{mm} \\
&= 100 \text{mm}^2.
\end{align*}
\]

In a similar way (not drawn to real size):

\[
\begin{align*}
1 \text{m}^2 &= 100 \text{cm} \times 100 \text{cm} \\
&= 10000 \text{cm}^2.
\end{align*}
\]

**9. Compound Areas Composed of Rectangles (NNS Year 6)**

After the area of a rectangle formula has been established, pupils can be taught to apply this to non-rectangular areas which can be subdivided into rectangles. For example, the area of the awkward shape below could be subdivided into rectangles like this and the total area found by adding the areas of all the rectangles. A more sophisticated approach would be to subdivide the shape like this and then calculate the total area as (area of the 3 large squares) - (the small rectangles where they overlap).
10. Surface Areas of Cuboids (NNS Year 6)
The surface area of a 3D shape can be defined as the sum of the areas of its faces. Since the faces of cuboids are rectangles, the area of a rectangle formula can be applied to each face and the resulting areas added to find the total surface area. Children will need to be shown that the areas of opposite faces of a cuboid are equal and so only the areas of 3 faces need to be calculated using the rectangle area formula. Thus for the cuboid below:

\[
\text{the total surface area} = (2 \times \text{area of face A}) + (2 \times \text{area of face B}) + (2 \times \text{area of face C}).
\]

Further Development in Area
The following further material is included for work with children of higher attainment and for subject knowledge purposes.

1. The Area of a Triangle
Deriving the rectangle formula above provides us with a powerful tool with which to arrive at area formulae for other shapes. The formula for the area of the triangle is usually tackled next. The technique is to construct a rectangle around the triangle so that the area of the rectangle is twice that of the triangle.

So, for any right-angled triangle

we construct the dotted-line rectangle

The area of the rectangle = \(l \times b\) (using the area of a rectangle formula) and the area of the triangle is half of the area of the rectangle.

For any non-right-angled triangle

we construct the dotted-line rectangle

and subdivide the triangle to create two smaller triangles:

It can be seen then that the two parts of the original triangle are equal in area to the remainder of the rectangle. Thus, the area of the triangle is half of the area of the rectangle.

Since all triangles can be treated in this way, we arrive at

\[
\text{Area of a triangle} = \frac{1}{2} b \times h, \]

where \(b\) = length of the base and \(h\) = perpendicular height of the triangle.
2. Area of a Parallelogram
Once again, the area of a rectangle formula can be used to derive a corresponding formula for parallelograms. The strategy uses the conservation of area idea. Any parallelogram can be cut along the dotted line and the end triangle repositioned at the opposite end of the parallelogram to form a rectangle with the same perpendicular height \( h \) and length of base \( b \) as the original parallelogram (using conservation of area idea). Thus, using the formula, the area of this equivalent rectangle = \( b \times h \), so

\[
\text{the area of a parallelogram} = b \times h.
\]

3. Area of a Trapezium
Deriving the area of a trapezium formula is less obvious. The strategy is to divide any trapezium into a parallelogram and a triangle (for which area formulae have already been derived):

The area of the parallelogram = \( a \times h \) and
the area of the triangle = \( \frac{1}{2} \times (b - a) \times h \) so, adding these together we obtain
the area of the trapezium = \( ah + \frac{1}{2}(b - a)h \)
\[= ah + \frac{1}{2}bh - \frac{1}{2}ah \quad \text{(removing the bracket)}\]
\[= \frac{1}{2}ah + \frac{1}{2}bh \quad \text{(combining like terms)}\]
\[= \frac{1}{2}(a + b)h \quad \text{(finding common factors \( \frac{1}{2} \) and \( h \))}\]

\[
The \text{area of a trapezium} = \frac{1}{2}(a + b) \times h.
\]

4. The Area of a Circle
Derivation of the formula for the area of a circle relies on the conservation of area idea and uses both the formula for the area of a rectangle and also that for the circumference of a circle (see Length and Perimeter booklet). Any circle is cut into lots of equal sectors like this (a small number of sectors is used here for visual clarity):

These sectors are then rearranged to form an approximate ‘rectangle’ (with one of the sectors being bisected in order to form the ends of the ‘rectangle’):
The height of the ‘rectangle’ is the same as the radius \( r \) of the circle. The circumference \( C \) of the circle has been dissected into lots of arcs, half of which form the top of the ‘rectangle’ and and half of which form the bottom of the ‘rectangle’.

Thus, the length of the ‘rectangle’ is equal to half of the circumference \( C \) of the circle.

Given that the circumference of a circle = \( 2\pi r \), it can be deduced that half of the circumference = \( \pi r \).

Thus, the ‘rectangle’ has dimensions \( \pi r \times r \): 

If the original circle was divided into sectors which were infinitely small (i.e. cut up the circle into very many thin sectors) then the humps along the length of the rectangle would become a straight line. So, using very thin sectors we can make a rectangle.

Then, using the area of a rectangle formula, we find that

The area of the ‘rectangle’ = length \( \times \) breadth

\[ = \pi r \times r \]

\[ = \pi r^2 \]

But the area of the ‘rectangle’ is equal to the area of the sectors and, therefore, to the area of the original circle. So,

\[ \text{the area of the circle} = \pi r^2. \]

5. Area of a Sector

A sector is a fraction of a circle so its area is a fraction of \( \pi r^2 \).

For example, this 45° sector is \( \frac{45}{360} \) of a circle so its area = \( \frac{45}{360} \times \pi r^2 \).
Common Errors and Misconceptions

1. Failure to Understand Conservation of Area
   The child believes that rearrangement of sections of a given area changes the total area. For example, the child believes that cutting this shape along the dotted line and rearranging the resulting pieces to give changes the area. Children will often say that the parallelogram has a greater area because “it’s more spread out”.

2. Over-Generalisation of the Area of a Rectangle Formula
   Having been introduced to the formula for the area of a rectangle the child believes it applies to areas of all shapes e.g. calculating the area of a leaf by measuring its length and width and multiplying them.

3. Confusing Area and Perimeter
   This is a very common difficulty. The child confuses the definitions because both are concerned with the boundary of a shape, perimeter being the distance around it and area being the amount of surface within it. For example, for the rectangle the child records the area as 20cm. This is particularly common when no visual marking of the units for area (e.g. a square centimetre grid) is apparent.

4. Belief in Area and Perimeter Interdependence
   The child believes incorrectly that altering the perimeter of a shape must change the area or that altering the area must change the perimeter. Of course, the truth is that changing either the area or the perimeter may or may not change the other i.e. there is no direct relationship.

5. Use of Inappropriate Units
   The child calculates (using an area formula) or measures the area correctly but either supplies no units or incorrect units. For example, for the area of the rectangle shown above common responses would be 24, 24 cm, and 24 cm$^3$.

6. Poor Understanding of the Relationship between Length and Area
   The child incorrectly believes that multiplying the length of the sides of a 2D shape results in the area being multiplied by the same scale factor. e.g. the child thinks that doubling the length of each of the sides of the rectangle below (which has an area of 6cm$^2$)

Area 12 © 2000 Andrew Harris
2 cm \[ \text{will result in a rectangle of area } 12 \text{ cm}^2. \]

3 cm

7 Poor Understanding of the Relationship between Surface Area and Volume
This is similar to that above for area and length. The child believes (incorrectly) that multiplying the surface areas of each of the faces results in the volume being multiplied by the same scale factor.

Relationships between Metric Units: Area

- \(100 \text{ mm}^2 = 1 \text{ cm}^2\)
- \(10,000 \text{ cm}^2 = 1 \text{ m}^2\)
- \(100 \text{ m}^2 = 1 \text{ are}\)
- \(10,000 \text{ m}^2 = 1 \text{ hectare}\)
- \(1,000,000 \text{ m}^2 = 1 \text{ km}^2\)

Useful Resources for Teaching Area

The tangram is useful for developing children’s understanding of the conservation of area.

Using a Tangram to Teach Area
The tangram should be cut up into its constituent shapes. It is best mounted or photocopied onto card.

Some possible activities using the tangram overleaf include:

- Use the smallest triangle as a non-standard unit. Measure each of the other shapes in terms of this triangle by direct comparison or using relevant area formulae;
- Find different combinations of some of the shapes to make an area of the same size as the big triangle;
- Make some different pentagons/hexagons etc. which have the same area;
- Make some area statements relating different shapes e.g. the area of the parallelogram is twice as much as the area of the smallest triangle. Also make the inverse statement i.e. the area of the smallest triangle is half that of the parallelogram;
- Make a shape using the individual pieces (or a subset of them), draw around the outline and then rearrange the pieces to produce a new shape. Compare the areas of the new and old arrangements.

Tangrams do not have to be square. Try making your own. Start with any shape and find a way to dissect it into other smaller shapes.

The Geoboard papers are useful for recording practical work undertaken on geoboards. The 1 cm\(^2\) squared paper grid can be photocopied onto acetates for use as an overlay with which to measure areas by counting 1 cm\(^2\) squares.
Square Tangram